Russian Options for a Diffusion with Negative Jumps

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February 5, 2001

Abstract

Closed solutions to the problem of pricing a Russian option when the underlying process is a diffusion with negative jumps are obtained. More precisely, the underlying process is assumed to have the form of a Wiener process with drift and negative mixed–exponentially distributed jumps driven by a Poisson process. This results generalize those of Shepp and Shiryaev (1993) for the Wiener process and Gerber, Michaud and Shiu (1995) for pure–jumps process.

1 Introduction and main results

1.1 Consider a model of a financial market with two assets, a savings account $B = (B_t)_{t\geq 0}$, and a stock $S = (S_t)_{t\geq 0}$. The evolution of B is deterministic, with

$$B_t = B_0 e^{rt}; \quad B_0 > 0, \quad r > 0,$$

and the stock is random, and evolves according to the formula

$$S_t = S_0 e^{X_t}; \quad S_0 > 0, \tag{1}$$

where $X = (X_t)_{t\geq 0}$ is a stochastic process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, a stochastic basis that satisfy the usual conditions. Consider also the supremum process, denoted by $(S_t^*)_{t\geq 0}$, and given by

$$S_t^* = \sup_{0\leq r\leq t} S_r.$$

In this model L. Shepp and A. N. Shiryaev [SS93] introduced an American option type on the maximum value of the stock, baptized as *Russian option*. Related to this, we mention the European options on the maximum introduced by A. Conze and Viswanathan [CV91], called *look–back options*. In [SS93] and [SS94] closed solutions were obtained for the problem of pricing Russian options in the perpetual case, in the framework of the Black–Scholes–Merton (1973) model (see [BS73]), this is to say, when X is a Wiener process with drift. Afterwards, Gerber, Michaud and Shiu, in [GMS95] gave closed solutions to prices of perpetual Russian options when the underlying process was a *risk process*, more precisely, a compound Poisson process with mixed exponentially distributed negative jumps and deterministic drift.

1.2 The purpose of the present paper is to unify these results, that is, to give closed solutions to the following optimal stopping problem.

- The process X in the stock (1) has the form

$$X_t = \Big(a - \frac{\sigma^2}{2}\Big)t + \sigma W_t - \sum_{i=1}^{N_t} Y_i, \tag{2}$$

  where $W = (W_t)_{t\geq 0}$ is a standard Wiener process, $\sigma > 0$, $N = (N_t)_{t\geq 0}$ is a Poisson process with intensity c, and $Y = (Y_k)_{k\in\mathbb{N}}$ is a sequence of non–negative independent random variables with common distribution

$$F(y) = 1 - \sum_{i=1}^{n} A_i e^{-\alpha_i y}, \quad y \geq 0, \tag{3}$$

  where $A_i > 0$ for $i = 1, 2, \dots, n$; $\sum_{i=1}^n A_i = 1$; and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. The processes W, N and Y are independent.

- The payoff $(f_t)_{t\geq 0}$ of the perpetual American option takes the form

$$f_t = e^{-\lambda t} \max\big[S_t^*, S_0\psi_0\big]$$

  with $\lambda \geq 0$ a discount factor and $\psi_0 \geq 1$.

To price this contract we can assume that

$$r = a + c\int_0^{+\infty} (e^{-y} - 1)\, dF(y),$$

and this implies that P is a martingale measure. Anyhow, we consider a more general situation, introducing a dividend rate ρ, given by

$$\rho = r - a - c\int_0^{+\infty} (e^{-y} - 1)\, dF(y) \tag{4}$$

under the restriction $\rho \geq 0$. With this assumptions the process $(e^{(\rho-r)t}S_t)_{t\geq 0}$ is a martingale under P.

Rational pricing of Russian options in complete markets led to the consideration of an optimal stopping problem. We solve the following question: find a function $C(\psi_0)$ and a stopping time τ^* such that

$$C(\psi_0) = \sup_{\tau\in\mathcal{M}} \mathsf{E}\, e^{-(\lambda+r)\tau} \max\big[S_\tau^*, S_0\psi_0\big] = \mathsf{E}\, e^{-(\lambda+r)\tau^*} \max\big[S_{\tau^*}^*, S_0\psi_0\big] \tag{5}$$

where $\mathcal{M}$ is the class of all P–finite stopping times.

1.3 Dual Martingale measure. In the case considered, according to (2), X is a Lévy process. If $q \in \mathbb{R}$, Lévy–Khinchine's formula states

$$\mathsf{E}\, e^{iqX_t} = \exp\Big\{ t\Big[\Big(a - \frac{\sigma^2}{2}\Big) iq - \frac{\sigma^2}{2} q^2 + c \int_{\mathbb{R}} (e^{iqx} - 1)\, dF(x) \Big] \Big\}. \tag{6}$$

Taking into account (3), if $z \in \mathbb{C}$ with $\mathsf{Re}(z) > -\alpha_1$, the characteristic exponent $\Psi = \Psi(z)$ defined through

$$\mathsf{E}\, e^{zX_t} = e^{t\Psi(z)},$$

completely determines the law of X, and takes the form

$$\begin{aligned} \Psi(z) &= \Big(a - \frac{\sigma^2}{2}\Big) z + \frac{\sigma^2}{2} z^2 + c \int_0^{+\infty} (e^{-zy} - 1)\, dF(y) \\ &= \Big(a - \frac{\sigma^2}{2}\Big) z + \frac{\sigma^2}{2} z^2 - c \sum_{i=1}^{n} A_i \frac{z}{z + \alpha_i}. \end{aligned} \tag{7}$$

Our path–dependent problem is transformed into an optimal stopping problem of a Markov process through a *change of numeraire*, that corresponds to a change of measure, leading to the introduction of the *dual martingale measure*. This procedure was introduced in [SKKM94, SS94, KM94]. In Proposition 1 we construct the measure $\widetilde{\mathsf{P}}$ and show, that under this new probability measure, X is a Lévy process with characteristic exponent

$$\widetilde{\Psi}(z) = \widetilde{a} z + \frac{\sigma^2}{2} z^2 - \widetilde{c} \sum_{i=1}^{n} \widetilde{A}_i \frac{z}{z + \widetilde{\alpha}_i}. \tag{8}$$

The *dual* parameters are given by Girsanov's Theorem,

$$\widetilde{a} = a + \sigma^2/2, \qquad \widetilde{c}\, \widetilde{F}(dy) = e^{-y} c\, F(dy). \tag{9}$$

This gives that under $\widetilde{\mathsf{P}}$ the process X changes its distribution only trough its parameters, according to

$$\widetilde{c} = c \sum_{i=1}^{n} \frac{A_i \alpha_i}{1 + \alpha_i}, \quad \widetilde{\alpha}_i = \alpha_i + 1, \quad \widetilde{A}_i = \frac{A_i \alpha_i}{1 + \alpha_i} \Big/ \sum_{i=1}^{n} \frac{A_i \alpha_i}{1 + \alpha_i}, \tag{10}$$

for $i = 1, \dots, n$. We denote also by $\widetilde{\Psi}$ the analytical continuation of the characteristic exponent of X under $\widetilde{\mathsf{P}}$.

1.4 Main Result. We are in position to formulate our main result.

THEOREM 1 *Consider the market model in 1.1. Assume that ρ in* (4) *satisfies $\rho \geq 0$. Then, the solution to the optimal stopping problem* (5) *for $\psi_0 \geq 1$ has cost function*

$$C(\psi_0) = S_0 \begin{cases} \widetilde{\psi} \Big[C_0 \Big(\frac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_0} + \cdots + C_{n+1} \Big(\frac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_{n+1}} \Big] & \text{if } 1 \leq \psi_0 < \widetilde{\psi} \\ \psi_0 & \text{if } \widetilde{\psi} \leq \psi_0, \end{cases} \tag{11}$$

where $\beta_0, \ldots, \beta_{n+1}$ *are the real roots of the equation*

$$\widetilde{\Psi}(-\beta) = \lambda + \rho, \tag{12}$$

with $\widetilde{\Psi}$ *defined in* (8), *and satisfy*

$$\beta_0 < 0 < 1 < \beta_1 < \alpha_1 + 1 < \cdots < \beta_n < \alpha_n + 1 < \beta_{n+1}. \tag{13}$$

Coefficients $C_0, \ldots, C_{n+1}$ *are given by*

$$C_i = \prod_{k=1}^{n} \left(\frac{\alpha_k + 1 - \beta_i}{\alpha_k} \right) \prod_{\substack{k=0 \\ k \neq i}}^{n+1} \left(\frac{\beta_k - 1}{\beta_k - \beta_i} \right),$$

and $\widetilde{\psi} > 1$ *is the only root of the equation in* ψ

$$\beta_0 C_0 \psi^{-\beta_0} + \cdots + \beta_{n+1} C_{n+1} \psi^{-\beta_{n+1}} = 0. \tag{14}$$

The optimal stopping time is

$$\tau^* = \inf\left\{ t \geq 0 \colon \frac{\max\left[S_t^*, S_0\psi_0\right]}{S_t} \geq \widetilde{\psi} \right\} \tag{15}$$

and it is P*–a.s. finite.*

2 Proof

The first step of the proof consist in a change of numeraire that led us to the solution of a different optimal stopping problem, having the advantage that the underlying process is not path–dependent. The second part is the solution of the deterministic free boundary problem for an integro–differential operator, related to the generator of this auxiliary process.

Let us introduce a probability measure $\widetilde{\mathsf{P}}$ on $(\Omega, \mathcal{F})$ by its restrictions to $\mathcal{F}_t$, as

$$\frac{d\widetilde{\mathsf{P}}_t}{d\mathsf{P}_t} = e^{\rho t} \frac{B_0 S_t}{S_0 B_t}, \tag{16}$$

and stochastic processes $(M_t)_{t \geq 0}$ and $(\psi_t)_{t \geq 0}$ by

$$M_t = \max[S_t^*, S_0\psi_0], \qquad \psi_t = \frac{M_t}{S_t}. \tag{17}$$

Proposition 1 *(a) There exists a probability measure* $\widetilde{\mathsf{P}}$ *such that* $\widetilde{\mathsf{P}}|_{\mathcal{F}_t} = \widetilde{\mathsf{P}}_t$ *with* $\widetilde{\mathsf{P}}_t$ *defined in* (16).

(b) Under $\widetilde{\mathsf{P}}$, the process X is a Lévy process with characteristic exponent

$$\widetilde{\Psi}(iu) = i\widetilde{a}u - \frac{\sigma^2}{2}u^2 + \widetilde{c}\int_0^{+\infty}(e^{-iux}-1)\,d\widetilde{F}(x)$$

for real u, with

$$\widetilde{a} = a + \sigma^2/2, \qquad \widetilde{c}\,\widetilde{F}(dy) = e^{-y}c\,F(dy).$$

(c) If $\widetilde{\mathsf{E}}$ denotes expectation with respect to $\widetilde{\mathsf{P}}$, for an arbitrary bounded stopping time τ we have

$$\mathsf{E}\,e^{-(\lambda+r)\tau}M_\tau = S_0\,\widetilde{\mathsf{E}}\,e^{-(\lambda+\rho)\tau}\psi_\tau. \tag{18}$$

In view of (c) in the previous Proposition, we must solve an optimal stopping problem under $\widetilde{\mathsf{P}}$ for the process $(\psi_t)_{t\geq 0}$. Consider then the infinitesimal generator of ψ, given by

$$L^\psi f(z) = -azf'(z) + \frac{\sigma^2}{2}z^2 f''(z) + \widetilde{c}\int_0^{+\infty}\big[f(ze^x) - f(z)\big]\,d\widetilde{F}(x).$$

In case f is only once differentiable and convex, by f'' we mean the second derivative from the left. The way to find the solution to this associated optimal stopping problem under $\widetilde{\mathsf{P}}$ is solving the free–boundary problem, consisting in finding a constant $\widetilde{\psi} > 1$ and a real function $V = V(\psi)$ with $\psi \geq 1$ such that

$$\begin{cases} L^\psi V(z) - (\lambda+\rho)V(z) = 0 & \text{if } 1 \leq z \leq \widetilde{\psi}, \\ V(\widetilde{\psi}) = \widetilde{\psi}, \\ V'(1+) = 0, \\ V'(\widetilde{\psi}-) = 1. \end{cases} \tag{19}$$

The next proposition presents some technical results, while Propositions 3 and 4 contain the key information to solve this problem.

Proposition 2 *(a) The equation in β given by*

$$\widetilde{\Psi}(-\beta) = \lambda + \rho \tag{20}$$

has $n+2$ roots $\beta_0, \beta_1, \dots, \beta_{n+1}$, that satisfy

$$\beta_0 < 0 < 1 < \beta_1 < \alpha_1 < \dots < \beta_n < \alpha_n + 1 < \beta_{n+1}. \tag{21}$$

(b) Coefficients C_i in Theorem 1 satisfy the following system of linear equations

$$\sum_{i=0}^{n+1} C_i \frac{1}{\widetilde{\alpha}_k - \beta_i} = \frac{1}{\widetilde{\alpha}_k - 1}, \quad \text{for } k = 1, \dots, n; \tag{22}$$

$$\sum_{i=0}^{n+1} \beta_i C_i = 1; \tag{23}$$

$$\sum_{i=0}^{n+1} C_i = 1; \tag{24}$$

with $\widetilde{\alpha}_k = \alpha_k + 1$*. Furthermore,* $C_i > 0$ *for* $i = 0, 1, \dots, n+1$*.*

(c) The function

$$f(x) = \beta_0 C_0 x^{-\beta_0} + \dots + \beta_{n+1} C_{n+1} x^{-\beta_{n+1}}, \quad x > 0, \tag{25}$$

has only one root $\widetilde{\psi} > 1$*.*

The following proposition gives the solution to the free boundary problem.

PROPOSITION 3 *Consider a function* V *defined by*

$$V(\psi_0) = \begin{cases} \widetilde{\psi}\Big[C_0\Big(\dfrac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_0} + \dots + C_{n+1}\Big(\dfrac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_{n+1}}\Big] & \text{if } 1 \le \psi_0 < \widetilde{\psi} \\ \psi_0 & \text{if } \widetilde{\psi} \le \psi_0, \end{cases}$$

Then, the following holds:

(a) The function V *is convex, continuously differentiable for all* $\psi \ge 1$ *and twice differentiable for all* $\psi \ne \widetilde{\psi}$*.*

(b) For all $z \ge 1$

$$L^{\psi} V(z) - (\lambda + \rho) V(z) \le 0.$$

(c) Furthermore, if $1 \le z \le \widetilde{\psi}$*, then*

$$L^{\psi} V(z) - (\lambda + \rho) V(z) = 0.$$

PROPOSITION 4 *For the function* V *and the process* $\psi = (\psi_t)_{t \ge 0}$ *as above,*

$$e^{-(\lambda+\rho)t} V(\psi_t) - V(\psi_0) = \int_0^t e^{-(\lambda+\rho)s} \big[L^{\psi} V(\psi_{s^-}) - (\lambda+\rho) V(\psi_{s^-})\big]\, ds + Q_s \tag{26}$$

for all $t \ge 0$*, where* $(Q_t)_{t \ge 0}$ *is a local martingale under* $\widetilde{\mathsf{P}}$*.*

PROOF (*of the Theorem*): We verify the following two assertions for the function $C(\psi_0)$ in (11). Observe that $C(\psi_0) = S_0 V(\psi_0)$.

(a) $\mathsf{E}\, e^{-(\lambda+r)\tau} M_\tau \le C(\psi_0)$, for any $\tau \in \mathcal{M}$;

(b) $\mathsf{E}\, e^{-(\lambda+r)\tau^*} M_{\tau^*} = C(\psi_0)$, for τ^* defined in (15).

Let us verify (a). Take $\tau \in \mathcal{M}$; by Proposition 4 and (b) in Proposition 3 we have

$$e^{-(\lambda+\rho)\tau\wedge t} V(\psi_{\tau\wedge t}) - V(\psi_0) \le Q_{\tau\wedge t}, \tag{27}$$

so $(Q_{\tau\wedge t})_{t\ge 0}$ is a supermartingale. As $Q_0 = 0$, $\widetilde{\mathsf{P}}$–expectations in (27) give $\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau\wedge t} V(\psi_{\tau\wedge t}) \le V(\psi_0)$. So

$$\begin{aligned} \mathsf{E}\, e^{-(\lambda+r)\tau\wedge t} M_{\tau\wedge t} &= S_0\, \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau\wedge t} \psi_{\tau\wedge t} \\ &\le S_0\, \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau\wedge t} V(\psi_{\tau\wedge t}) \le S_0 V(\psi_0). \end{aligned} \tag{28}$$

Now, by Fatou's Lemma, as $\mathsf{P}(\tau < \infty) = 1$ we have

$$\mathsf{E}\, e^{-(\lambda+r)\tau} M_\tau \le \lim_{t\to\infty} \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau\wedge t} \psi_{\tau\wedge t}$$

and (a) follows. In order to prove (b), we verify that $(Q_{\tau^*\wedge t})_{t\ge 0}$ is an uniform integrable $\widetilde{\mathsf{P}}$–martingale. By Proposition 4 and (c) in Proposition 3, as $\psi_{\tau^*\wedge t^-} \le \widetilde{\psi}$, we have

$$e^{-(\lambda+\rho)\tau^*\wedge t} V(\psi_{\tau^*\wedge t}) - V(\psi_0) = Q_{\tau^*\wedge t}. \tag{29}$$

Therefore

$$\begin{aligned} -V(\psi_0) \le Q_{\tau^*\wedge t} &\le e^{-(\lambda+\rho)\tau^*\wedge t} V(\psi_{\tau^*\wedge t}) \\ &= e^{-(\lambda+\rho)t} V(\psi_t) \mathbb{I}_{\{t<\tau^*\}} + e^{-(\lambda+\rho)\tau^*} V(\psi_{\tau^*}) \mathbb{I}_{\{\tau^*\le t\}} \\ &\le V(\widetilde{\psi}) + e^{-(\lambda+\rho)\tau^*} \psi_{\tau^*}. \end{aligned}$$

To conclude the uniform integrability of $(Q_{\tau^*\wedge t})_{t\ge 0}$ it is enough to see that $e^{-(\lambda+\rho)\tau^*} \psi_{\tau^*}$ has finite $\widetilde{\mathsf{P}}$ expectation. First observe that $\widetilde{\mathsf{P}}(\tau^* < \infty) = 1$. This follows based on the property of homogeneous independent increments of X, as done in [SS94], see also [Mor00]. By Fatou's Lemma and (28),

$$\begin{aligned} \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*} \psi_{\tau^*} &= \widetilde{\mathsf{E}} \Big[\lim_{t\to+\infty} e^{-(\lambda+\rho)\tau^*\wedge t} \psi_{\tau^*\wedge t} \Big] \\ &\le \liminf_{t\to+\infty} \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*\wedge t} \psi_{\tau^*\wedge t} \le V(\psi_0) \end{aligned}$$

as τ^* is $\widetilde{\mathsf{P}}$–finite. Now, we have $\widetilde{\mathsf{E}}(Q_{\tau^*}) = 0$ and thus, by (29),

$$\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*\wedge t} \psi_{\tau^*\wedge t} \longrightarrow \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*} \psi_{\tau^*} = \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau^*} V(\psi_{\tau^*}) = V(\psi_0).$$

On the other hand

$$\begin{aligned} \mathsf{E}\, e^{-(\lambda+r)\tau^*\wedge t} M_{\tau^*\wedge t} &= \mathsf{E}\, e^{-(\lambda+r)t} M_t \mathbb{I}_{\{t<\tau^*\}} + \mathsf{E}\, e^{-(\lambda+r)\tau^*} M_{\tau^*} \mathbb{I}_{\{\tau^*\le t\}} \\ &= \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)t} \psi_t \mathbb{I}_{\{t<\tau^*\}} + \mathsf{E}\, e^{-(\lambda+r)\tau^*} M_{\tau^*} \mathbb{I}_{\{\tau^*\le t\}} \\ &\to \mathsf{E}\, e^{-(\lambda+r)\tau^*} M_{\tau^*}. \end{aligned}$$

as $t \to +\infty$, since $\widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)t}\psi_t \mathbb{I}_{\{t<\tau^*\}}$ is bounded by $\widetilde{\psi}\,\widetilde{\mathsf{P}}(t < \tau^*)$ and τ^* is $\widetilde{\mathsf{P}}$–finite. Then, part (b) follows from part (c) of proposition 1. This concludes the proof of the Theorem. □

3 Appendix: Proof of Propositions

Proof (*of Proposition 1*): For the part (a), since $Z_t = e^{\rho t} B_0 S_t / S_0 B_t$ is a martingale, the construction of $\widetilde{\mathsf{P}}$ follows as in §1.3 in [SKKM94].

For the part (b) we compute the characteristic exponent of X under $\widetilde{\mathsf{P}}$. For $u \in \mathbb{R}$ we have

$$\widetilde{\mathsf{E}}\, e^{iuX_t} = \mathsf{E}\Big(e^{iuX_t} e^{\rho t} \frac{B_0 S_t}{S_0 B_t}\Big) = \mathsf{E} \exp\big[(iu+1)X_t + \rho t - rt\big] \\ = \exp\big[t(\Psi(iu+1) + \rho - r)\big],$$

with Ψ as in (7). Now, taking into account (4):

$$\Psi(iu+1) + \rho - r = \Big(a - \frac{\sigma^2}{2}\Big)(iu+1) + \frac{\sigma^2}{2}(iu+1)^2 \\ + c\int_0^{+\infty} (e^{-(iu+1)x} - 1)\, dF(x) \\ = \Big(a + \frac{\sigma^2}{2}\Big) iu - \frac{\sigma^2}{2}u^2 + \widetilde{c}\int_0^{+\infty} (e^{-iux} - 1)\, d\widetilde{F}(x),$$

proving (b).

Now we prove (c). Measures $\widetilde{\mathsf{P}}$ and P are locally mutually absolutely continuous, with density process $Z = (Z_t)_{t\geq 0}$ given by $Z_t = e^{\rho t} B_0 S_t / S_0 B_t$. When τ is bounded, by III.3.4 in [JS87],

$$\mathsf{E}\, e^{-(\lambda+r)\tau} M_\tau = \mathsf{E}\Big(e^{\rho\tau} \frac{B_0 S_\tau}{S_0 B_\tau} \times \frac{S_0 e^{-(\lambda+\rho)\tau} M_{\tau\wedge t}}{S_\tau}\Big) \\ = S_0\, \widetilde{\mathsf{E}}\, e^{-(\lambda+\rho)\tau}\psi_\tau.$$

concluding the proof. □

Proof (*of Proposition 2*): Let us prove (a). Taking into account (8), (9) and (10),

$$\widetilde{\Psi}(-\beta) = -\beta(a+\sigma^2) + \frac{\sigma^2}{2}\beta(\beta+1) + c\sum_{i=1}^n \frac{A_i\alpha_i}{1+\alpha_i-\beta} - c\sum_{i=1}^n \frac{A_i\alpha_i}{1+\alpha_i}.$$

So (20) reads

$$-\frac{\sigma^2}{2}\beta^2 + \Big(\frac{\sigma^2}{2} + a\Big)\beta + c\sum_{i=1}^n \frac{A_i\alpha_i}{1+\alpha_i} + \lambda + \rho = c\sum_{i=1}^n \frac{A_i\alpha_i}{1+\alpha_i-\beta}. \tag{30}$$

The roots are then given by the intersection of the graphs of a sum of n hyperbolae with a concave parabola. Evaluation at $\beta = 0$ gives that the parabola is bigger that the sum at this points, and the roots satisfy (21). In order to see $1 < \beta_1$ we evaluate both terms in (30) at $\beta = 1$ to see that at this point the parabola is bigger than the sum. For details see [Mor00].

To prove (b) we introduce two auxiliary polynomials

$$P(x) = \prod_{j=1}^{n} \big(1 + x/\alpha_j\big), \qquad Q(x) = \prod_{j=0}^{n+1} \big(1 + x/(\beta_j - 1)\big),$$

and consider the simple fractional expansion,

$$\frac{P(x)}{Q(x)} = \sum_{j=0}^{n+1} D_j \frac{1}{\beta_j - 1 + x}. \tag{31}$$

In order to determine the coefficients, as we have simple roots,

$$\begin{aligned} D_i &= \frac{P(1-\beta_i)}{Q'(1-\beta_i)} \\ &= \prod_{j=1}^{n} \Big(\frac{\alpha_j + 1 - \beta_i}{\alpha_j}\Big) \Bigg[\frac{1}{\beta_i - 1} \prod_{\substack{j=0 \\ j \neq i}}^{n+1} \Big(\frac{\beta_j - \beta_i}{\beta_j - 1}\Big)\Bigg]^{-1} = (\beta_i - 1) C_i. \end{aligned}$$

So, (31) becomes

$$\frac{P(x)}{Q(x)} = \sum_{j=0}^{n+1} C_i \frac{\beta_i - 1}{\beta_i - 1 + x}.$$

Now, taking $x = -\alpha_k$ for $k = 1, \dots, n$ and $x = 0$ in (31) we obtain (22) and (24) respectively. To see (23) we multiply both sides of (31) by x and take limits as $x \to \infty$, obtaining

$$\sum_{j=0}^{n+1} C_i(\beta_i - 1) = 0,$$

that in view of (24) concludes the proof. The properties $C_i > 0$ follows from (13).

For the part (c), as $C_i > 0$ for $i = 0, \dots, n+1$, by differentiation in (25) we get that f is decreasing, and $\lim_{x\to\infty} f(x) = -\infty$. We then see $f(1) > 0$. But

$$f(1) = \beta_0 C_0 + \cdots + \beta_{n+1} C_{n+1} = 1$$

in view of (23), proving the existence of a root bigger that one. $\square$

Proof (*of Proposition 3*): For the first part, clearly V is differentiable for all orders if $\psi \neq \widetilde{\psi}$. Equation (24) shows that $V(\widetilde{\psi}) = \widetilde{\psi}$ meaning that V is continuous, and equation (23) gives $V'(\widetilde{\psi}-) = 1$, showing that V is continuously differentiable, i.e. satisfies the *smooth pasting condition* (see [Shi78]). In what respects convexity, we examine the second derivative on $\psi_0 \in [1, \widetilde{\psi})$,

$$V''(\psi_0) = \widetilde{\psi} \sum_{i=0}^{n+1} \beta_i(\beta_i - 1) C_i \Big(\frac{\psi_0}{\widetilde{\psi}}\Big)^{\beta_i - 2} \geq 0,$$

because $C_i > 0$ and $\beta_i(\beta_i - 1) > 0$ in view of (21).

For the parts (b) and (c), take first $z > \widetilde{\psi}$. In this case $V(z) = z$ and $V(ze^x) = ze^x$ for $x \geq 0$. So $V''(z) = 0$ and

$$\begin{aligned} L^{\psi} V(z) - (\lambda + \rho) V(z) &= -az + \widetilde{c} \int_0^{+\infty} ze^x \, d\widetilde{F}(x) - z(\widetilde{c} + \lambda + \rho) \\ &= z\Big(-a + \widetilde{c} \sum_{i=1}^{n} \frac{\widetilde{A}_i \widetilde{\alpha}_i}{\widetilde{\alpha}_i + 1} - \widetilde{c} - \lambda - \rho\Big) \\ &= -z(r + \lambda) \leq 0 \end{aligned}$$

for all $z > \widetilde{\psi}$, where $\widetilde{c}$ and $\widetilde{A}$ are given in (10) and ρ in (4). Take now $\widetilde{\psi} \geq z$, so

$$\begin{aligned} L^{\psi} V(z) - (\lambda + \rho) V(z) = -az\widetilde{\psi} &+ \sum_{i=0}^{n+1} \beta_i C_i \Big(\frac{1}{\widetilde{\psi}}\Big) \Big(\frac{z}{\widetilde{\psi}}\Big)^{\beta_i - 1} \\ &+ \frac{\sigma^2}{2} z^2 \widetilde{\psi}^2 \sum_{i=0}^{n+1} \beta_i(\beta_i - 1) C_i \Big(\frac{1}{\widetilde{\psi}^2}\Big) \Big(\frac{z}{\widetilde{\psi}}\Big)^{\beta_i - 2} \\ &+ \widetilde{c}\widetilde{\psi} \int_0^{\log(\widetilde{\psi}/z)} \sum_{i=0}^{n+1} C_i \Big(\frac{z}{\widetilde{\psi}}\Big)^{\beta_i} e^{\beta_i x} \, d\widetilde{F}(x) \\ &+ \widetilde{c} \int_{\log(\widetilde{\psi}/z)}^{+\infty} ze^x \, d\widetilde{F}(x) - (\widetilde{c} + \lambda + \rho) \widetilde{\psi} \sum_{i=0}^{n+1} C_i \Big(\frac{z}{\widetilde{\psi}}\Big)^{\beta_i}, \end{aligned}$$

that, after computing the integrals became

$$\begin{aligned} &L^{\psi} V(z) - (\lambda + \rho) V(z) \\ &\quad = \sum_{i=0}^{n+1} \Big(\frac{z}{\widetilde{\psi}}\Big)^{\beta_i} C_i \widetilde{\psi} \Big\{-a\beta_i + \frac{\sigma^2}{2} \beta_i(\beta_i - 1) + \widetilde{c} \sum_{k=1}^{n} \frac{\widetilde{A}_k \widetilde{\alpha}_k}{\widetilde{\alpha}_k - \beta_i} - (\widetilde{c} + \lambda + \rho)\Big\} \\ &\qquad - \widetilde{c} \sum_{k=1}^{n} \widetilde{A}_k \widetilde{\alpha}_k \Big(\frac{z}{\widetilde{\psi}}\Big)^{\widetilde{\alpha}_k} \widetilde{\psi} \sum_{i=0}^{n+1} \Big\{\frac{C_i}{\widetilde{\alpha}_k - \beta_i} - \frac{1}{\widetilde{\alpha}_k - 1}\Big\}. \end{aligned}$$

By (30) the first sum is zero, and (22) annuls the second. In conclusion, $L^{\psi} V(z) - (\lambda + \rho) V(z) = 0$ if $z \leq \widetilde{\psi}$. □

Proof (*of Proposition 4*): First we use Itô's Formula to establish the following

$$V(\psi_t) - V(\psi_0) = \int_0^t L^\psi V(\psi_{s^-})\, ds + q_t \tag{32}$$

for all $t \geq 0$, where $(q_t)_{t\geq 0}$ is a local martingale. Let us apply Itô's–Meyer formula to the convex function V and process $(\psi_t)_{t\geq 0}$ as in Theorem 51 in [Pro90]:

$$\begin{aligned} V(\psi_t) - V(\psi_0) = &\int_0^t V'(\psi_{s^-})\, d\psi_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^a(\psi)\, \mu(da) \\ &+ \sum_{s\leq t}\big[V(\psi_s) - V(\psi_{s^-}) - V'(\psi_{s^-})\Delta\psi_s\big], \end{aligned} \tag{33}$$

where $L_t^a(\psi)$ is the local time of $(\psi_t)_{t\geq 0}$ at level a and μ is the second derivative of V in the sense of distributions. Due to the form of V we have $\mu(da) = V''(a)\, da$ with V'' the second derivative from the left. As V'' is bounded

$$\int_{-\infty}^{+\infty} L_t^a(\psi)\, \mu(da) = \int_{-\infty}^{+\infty} L_t^a(\psi) V''(a)\, da = \int_0^t V''(\psi_{s^-})\, d\langle \psi^c, \psi^c\rangle_s \tag{34}$$

by Corollary 1 to Theorem 51 in [Pro90]. In reference to $(\psi_t)_{t\geq 0}$, as $(M_t)_{t\geq 0}$ is continuous, then

$$d\psi_t = M_t dS_t{}^{-1} + S_t{}^{-1} dM_t. \tag{35}$$

Also, as $S_t{}^{-1} = S_0^{-1} e^{-X_t}$, we have

$$dS_t^{-1} = S_t^{-1}\left[-a\, dt - \sigma\, dW_t + d\Big(\sum_{s\leq t}(e^{-\Delta X_s} - 1)\Big)\right]. \tag{36}$$

So, (35) and (36) give

$$d\psi_t = \psi_{t^-}\left[-a\, dt - \sigma\, dW_t + d\Big(\sum_{s\leq t}(e^{-\Delta X_s} - 1)\Big)\right] + S_t^{-1}\, dM_t. \tag{37}$$

As $(M_t)_{t\geq 0}$ does not decrease, the last term in (37) is continuous with bounded variation, and

$$d\langle \psi^c, \psi^c\rangle_t = \sigma^2 \psi_{t^-}^2\, dt.$$

Let us now compensate the jump part in (33). From (37), we know $\Delta\psi_t = \psi_{t^-}(e^{-\Delta X_t} - 1)$, so

$$\begin{aligned} \sum_{s\leq t}\big[V(\psi_s) - V(\psi_{s^-})\big] &= \sum_{s\leq t}\big[V(\psi_{s^-} + \Delta\psi_s) - V(\psi_{s^-})\big] \\ &= \sum_{s\leq t}\big[V(\psi_{s^-} e^{-\Delta X_s}) - V(\psi_{s^-})\big] \\ &= \int_{\mathbb{R}\times[0,t]} \big[V(\psi_{s^-} e^{-x}) - V(\psi_{s^-})\big]\, \mu^X(dx, ds) \end{aligned}$$

where μ^X is the jump measure of the process X given by

$$\mu(\omega, dx, dt) = \sum_s \mathbb{I}_{\{\Delta X_s(\omega)\neq 0\}} \delta_{(\Delta X_s(\omega), s)}(dx, dt).$$

As μ^X is an extended Poisson measure, according to II.1.21 in [JS87], its compensator under $\widetilde{\mathsf{P}}$ is given by $\widetilde{c}\,\widetilde{F}(-dx)\times dt$. From this

$$\sum_{s\leq t}\big[V(\psi_s) - V(\psi_{s^-})\big] = \int_{\mathbb{R}\times[0,t]}\big[V(\psi_{s^-}e^x) - V(\psi_{s^-})\big]\widetilde{c}\, ds\, d\widetilde{F}(x) + q_t^1$$

with $(q_t^1)_{t\geq 0}$ a local martingale. All this computations and (33) give

$$\begin{aligned} V(\psi_t) - V(\psi_0) = \int_0^t \Big\{ & -a\psi_{s^-}V'(\psi_{s^-}) + \frac{\sigma^2}{2}\psi_{s^-}^2 V''(\psi_{s^-}) \\ & + \widetilde{c}\int_0^{+\infty}\big[V(\psi_{s^-}e^x) - V(\psi_{s^-})\big]\, d\widetilde{F}(x)\Big\}\, ds \\ & + q_t^1 + \int_0^t \psi_{s^-}V'(\psi_{s^-})(-\sigma)\, dW_s + \int_0^t \frac{V'(\psi_{s^-})}{S_s}\, dM_s. \end{aligned} \tag{38}$$

As $\Delta X_t \leq 0$ the support of the measure dM_s is concentrated on the set where $M_t = S_t$, that is to say, when $\psi_s = 1$. But Proposition 2 gives $V'(1) = 0$. So,

$$\int_0^t \frac{V'(\psi_{s^-})}{S_s}\, dM_s = \int_0^t \frac{V'(1)}{S_s}\mathbb{I}_{\{\psi_s = 1\}}\, dM_s = 0$$

since $V'(\psi_{s^-})\mathbb{I}_{\{\psi_s=1\}} = V'(1)\mathbb{I}_{\{\psi_s=1\}}$. If we define $q = (q_t)_{t\geq 0}$ by

$$q_t = q_t^1 + \int_0^t \psi_{s^-}V'(\psi_{s^-}), (-\sigma)\, dW_s$$

then (38) coincides with (32). The last step is the application of Itô's Formula to the process given by $\big(e^{-(\lambda+\rho)t}V(\psi_t)\big)_{t\geq 0}$. We obtain (26), where $(Q_t)_{t\geq 0}$ is a stochastic integral with respect to $(q_t)_{t\geq 0}$. □

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